Year 12 Mathematics

EAS 2.6

Algebraic Methods

Robert Lakeland & Carl Nugent

Contents

nuLake Ltd
Innovative Publisher of Mathematics Texts

Algebraic Methods 2.6

This achievement standard involves applying algebraic methods in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply algebraic methods in solving problems.	• Apply algebraic methods, using relational thinking, in solving problems.	• Apply algebraic methods, using extended abstract thinking, in solving problems.

- ◆ This achievement standard is derived from Level 7 of The New Zealand Curriculum and is related to the achievement objectives
  - ❖ manipulate rational, exponential, and logarithmic algebraic expressions
  - ❖ form and use linear and quadratic equations.

- ◆ Apply algebraic methods in solving problems involves:
  - ❖ selecting and using methods
  - ❖ demonstrating knowledge of algebraic concepts and terms
  - ❖ communicating using appropriate representations.

- ◆ Relational thinking involves one or more of:
  - ❖ selecting and carrying out a logical sequence of steps
  - ❖ connecting different concepts or representations
  - ❖ demonstrating understanding of concepts
  - ❖ forming and using a model;

  and also relating findings to a context, or communicating thinking using appropriate mathematical statements.

- ◆ Extended abstract thinking involves one or more of:
  - ❖ devising a strategy to investigate or solve a problem
  - ❖ identifying relevant concepts in context
  - ❖ developing a chain of logical reasoning, or proof
  - ❖ forming a generalisation;

  and also using correct mathematical statements, or communicating mathematical insight.

- ◆ Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.

- ◆ Methods include a selection from those related to:
  - ❖ manipulating algebraic expressions, including rational expressions
  - ❖ manipulating expressions with exponents, including fractional and negative exponents
  - ❖ determining the nature of the roots of a quadratic equation
  - ❖ solving exponential equations (which may include manipulating logarithms)
  - ❖ forming and solving linear and quadratic equations.

Factorising

Factorising

Rewriting an expression as a product of its factors is called factorising.

The expression $4x + 36$ could be written as $4(x + 9)$ because when we multiply it out we get $4x + 36$.

Factorising is the reverse procedure to expanding.

$$(x+1)(x-4) \underset{\text{Factorising}}{\overset{\text{Expanding}}{\rightleftharpoons}} x^2 - 3x - 4$$

We only need to factorise an expression if it consists of a sum (or difference) of two or more terms. We must then factorise all of the expression not just part of it.

We adopt different factorising techniques for different types of expressions.

Common Factors

When we study the expression below we see that all the terms that make it up have a factor in common, so the appropriate approach is to extract the highest common factor. For example, each term in the expression

$$4x^3y^2 + 6xy^5 - 20x^2y^7z$$

has 2, x, and y^2 as common factors. We identify these by first studying the numbers at the front (coefficients) and seeing that 2 is the highest common factor, then we study the x component (x^3, x and x^2) and note x is the highest common factor and finally the y component (y^2, y^5 and y^7) and note y^2 is the highest common factor.

We write these on the outside of the brackets and divide them into the component of each term.

$$4x^3y^2 + 6xy^5 - 20x^2y^7z = 2xy^2(2x^2 + 3y^3 - 10xy^5z)$$

2 divides into 4, 6 and 20 to give 2, 3 and 10.

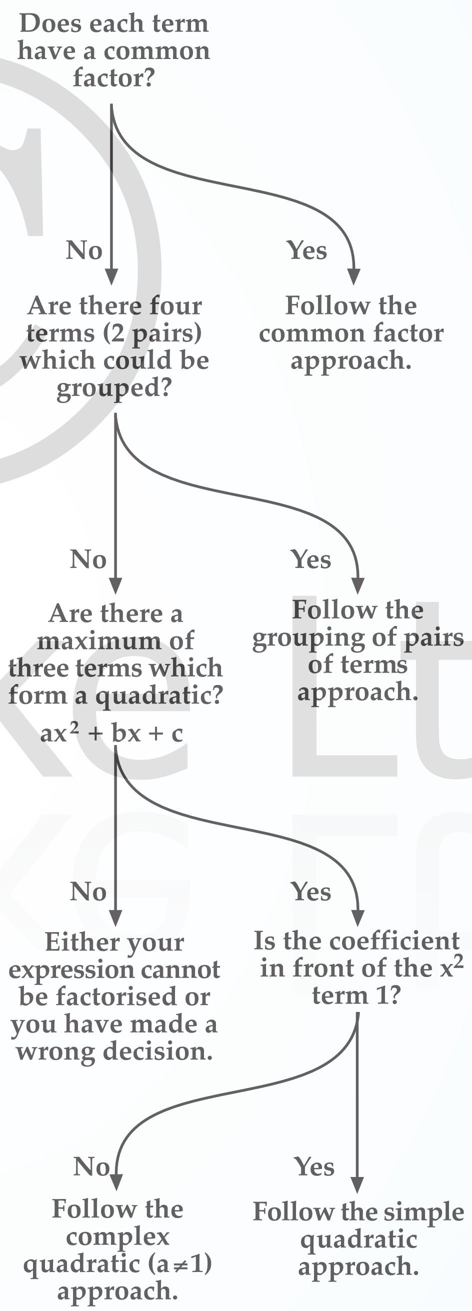
x divides into x^3, x and x^2 to give x^2, 1 and x.

y^2 divides into y^2, y^5 and y^7 to give 1, y^3 and y^5.

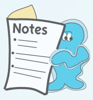
Our expression becomes

$$4x^3y^2 + 6xy^5 - 20x^2y^7z = 2xy^2(2x^2.1 + 3.1.y^3 - 10xy^5z)$$
$$= 2xy^2(2x^2 + 3y^3 - 10xy^5z)$$

There are four methods of factorising an expression. The flow chart below may be helpful in identifying the correct approach for a particular expression.



Logarithms



Logarithms

Logarithms are used when we wish to find the power a particular base (often but not always 10) is raised by to equal the number we are concerned with.

In general for base b

$$x = b^y$$

then $\quad \log_b x = y$

In base 10 the logarithm of the number 100 is the power 10 would be raised by to equal 100.

$$100 = 10^2$$

$$\log_{10} 100 = 2$$

On your calculator the $\boxed{\text{log}}$ button gives the base ten log.
If we find log 100 using the calculator we get the answer 2.

Similarly if we calculate log 3 using the calculator we get the answer 0.4771 as

$$3 = 10^{0.4771}$$

$$\log_{10} 3 = 0.4771$$

To check this out we find $10^{0.4771}$ on the calculator and to 4 significant figures it returns the answer 3.

Logs were introduced to remove the need for multiplication and division prior to the invention of calculators (see the notes on the right). We continue to use logarithms as they make the solution of some problems a lot easier.

We make use of the equivalent statement of logs in solving many problems.

$$y = b^x \text{ is equivalent to}$$

$$\log_b y = x$$

For example, to find what base two number has a log of 5 we substitute into the expression

$$\log_2 Z = 5$$

implies $\quad Z = 2^5$

$$Z = 32$$

If the base is not stated then by convention the base is taken as base 10.

$$\log N = \log_{10} N$$

A short history of Logarithms

Logarithms were invented by the Scottish mathematician John Napier.

He did so because they eliminated the need to multiply (or divide), as at the time all problems had to be done by hand.

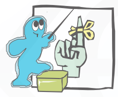
He used the principle that in multiplying the same base number to a power or exponent we add the power.

$$\begin{aligned} 8 \times 32 &= 2^3 \times 2^5 \\ &= 2^{3+5} \\ &= 2^8 \\ &= 256 \end{aligned}$$

A problem such as 23.45 x 456.2 is difficult and tedious by hand but if the numbers are able to be turned into powers of 10 the problem changes to addition.

$$\begin{aligned} 23.45 \times 456.2 &= 10^{1.370} \times 10^{2.659} \\ &= 10^{4.029} \\ &= 10\ 690 \text{ (4 sf)} \end{aligned}$$

Tables of logarithms (and the reverse antilogarithms) were produced so people could just look up the appropriate log and add them (or subtract for division). We continue to learn and use logarithms as they make many problems easier.



The base of a log must be positive and not equal to 1.

For $\quad \log_b y = x$

then $\quad y = b^x$

only if $\quad b > 0$ **and** $b \neq 1$.

Justification for b ≠ 1. If b = 1 then it implies

For $\quad \log_1 y = x$

then $\quad y = 1^x$

As $1^x = 1$ this equation has only a trivial solution of y = 1 for all x.

Justification for b > 1. If we have a base b < 0 and we define c = | b | then b = (⁻1) x c.

$$\begin{aligned} \log_b y &= x \\ y &= b^x \\ y &= ((^-1) \times c)^x \\ y &= (^-1)^x . c^x \end{aligned}$$

For whole number values of x the sign oscillates depending upon whether x is odd or even and when x is a fraction it implies we are taking a root of a negative number therefore we cannot have a log base that is negative.

Achievement – Use the equivalent log statement ($y = b^x$ is equivalent to $\log_b y = x$) to solve the following.

173. $\log_5 25 = A$

174. $\log_8 512 = B$

175. $\log_2 64 = C$

176. $\log_2 0.5 = D$

177. $\log_3 G = 3$

178. $\log_{10} H = 3$

179. $\log_M 32 = 5$

180. $\log_N 10 = 1$

181. $\log_P 1 = 0$

182. $\log_Q 243 = 5$

Merit – Solve the following.

183. $\log_2 (4y + 4) = 5$

184. $\log_3 (2x - 1) = 2$

185. $\log_5 \left(\frac{1}{x} + 23\right) = 2$

186. $\log_2 (x^2) = 6$

187. $\log_x 4(x - 1) = 2$

188. $\log_z (7z - 10) = 2$

Quadratic Equations

Quadratic Equations

If we are able to factorise a quadratic equation then finding the solutions is straightforward.

If we have a product of two factors that equal 0 then one (or both) of the factors must equal 0.

$$A.B = 0$$

then either $A = 0$ or $B = 0$

Therefore for a quadratic equation

$$(x+3)(x+2) = 0$$

then either $(x+2) = 0$

$$x = {}^-2$$

or $(x+3) = 0$

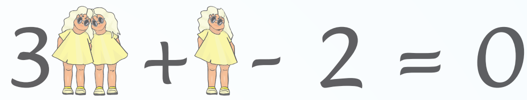
$$x = {}^-3$$

The solution is written as

$$x = {}^-3, {}^-2$$

If the quadratic is not already factorised, then we must factorise it for this method to work.

Be on the look out for quadratic equations disguised as something else.



A quadratic equation is a polynomial where the highest power of the unknown is two. A example is

$$3x^2 + x - 2 = 0$$
$$(3x-2)(x+1) = 0$$
$$x = {}^-1, \frac{2}{3}$$

but if we divide all terms by x and manipulate it the equation at first glance does not appear as a quadratic

$$3x + 1 = \frac{2}{x}$$

Yet this is an identical equation. Two other equations that can be solved as quadratic equations are

$$3x^4 + x^2 - 2 = 0$$

and

$$3x + \sqrt{x} = 2$$

In the first we substitute $z = x^2$ to get our quadratic and in the second we substitute $z = \sqrt{x}$.

Example

Solve the equation

$$2x^2 - x = 36$$

$$2x^2 - x = 36$$

Rearrange so that the equation equals zero first

$$2x^2 - x - 36 = 0$$

Factorising

$$(2x-9)(x+4) = 0$$

Setting each factor equal to 0 gives

$$(2x-9) = 0$$
$$x = 4.5$$

and $(x+4) = 0$

$$x = {}^-4$$
$$x = {}^-4, 4.5$$

Example

Solve the equation

$$3x + \sqrt{x} = 2$$

Substitute $z = \sqrt{x}$

$$3z^2 + z = 2$$

Rearrange so that the equation equals zero first

$$3z^2 + z - 2 = 0$$

Factorising

$$(3z-2)(z+1) = 0$$

Setting each factor equal to 0 gives

$$z = {}^-1, \frac{2}{3}$$
$$x = z^2$$

This would appear to give answers

$$x = 1, \frac{4}{9}$$

but substitution into the original equation we find the only answer is $x = \frac{4}{9}$

This is because the square root sign represents the positive root only (i.e. 1) which does not solve the original equation. We should always confirm these disguised questions by substituting back into the original equation.

407. $(p + 3)x^2 + 4x + p = 0$ has equal roots. Find the possible values of p.

408. What is the nature of the roots of $(q + 1)x^2 + (2q + 1)x + q = 0$?

409. If $(2k + 3)x^2 - 4kx + 4 = 0$ has equal roots find the value(s) of k.

410. Find the value(s) of k for which $x^2 + (k + 1)x = k + 1$ has equal roots.

411. If a, b and c are three consecutive terms of an arithmetic sequence show that the equation $(b - c)x^2 + (c - a)x + a - b = 0$ has equal roots. Use the discriminant and show all your working.

Practice External Assessment Task

Algebraic Methods 2.6

Make sure you show ALL relevant working for each question.

You are advised to spend 60 minutes answering this assessment.

QUESTION ONE

(a) Expand and simplify $(2x + 5)(x - 3)(x - 2)$

(b) Write as a single log statement $3\log A + \log B - 0.5 \log C$

(c) i) Solve the equation $3x^2 - 4x + 1 = 0$

ii) For what value of k does the equation $3x^2 + kx + 4 = 0$ have equal roots?

Answers

Page 4

1. $4x^2 + 20x$

2. $8x^2 - 12x$

3. $2x^3 + 3x^2$

4. $30a^2 - 18a$

5. $6k^3 + 4k^2 + 2k$

6. $12x^4 + 24x^3 - 28x^2$

7. $x^2 + 8x + 15$

8. $3x^2 + 2x - 1$

9. $6a^2 + 5a + 1$

10. $^-30x^2 + 25x + 30$

11. $15x^2 - 6x - 12$

12. $12x^2 - 9x + 3$

13. $x^2 - 2x - 12$

14. $16x^2 - 4x + 10$

15. $21x^2 - 14x$

16. $x^3 - 21x^2 + 147x - 343$

17. $12x^3 + 24x^2 - 3x - 6$

18. $3x^3 + x^2 - 8x + 4$

19. $a^3 + a^2 - 14a - 24$

20. $6x^3 + 13x^2 - 40x - 75$

21. $x^3 + x^2 - 8x - 12$

22. $6y^3 - 19y^2 + y + 6$

23. $x^3 - 11x^2 + 31x - 29$

24. $64x^3 + 48x^2 + 12x + 1$

25. $3k^3 - 16k^2 + 32k - 17$

26. $^-8y^3 + 12y^2 - 9y - 11$

27. $^-12x^2 - 16$

Page 5

28. $8q^3 - 12q^2 + 6q - 1$

29. $^-11m^2 + 5m$

30. $6p^3 - 7p^2 - 6p + 3$

31. $12x^2 - 28x + 8$

32. $m^3 - 7m - 6$

33. $24k^3 - 58k^2 + 23k + 15$

34. a) $2x + 40$

b) $\pi x^2 + 40\pi x + 400\pi$

c) $40\pi x + 400\pi$

d) $x = 5.92$

Diameter of stage is 11.8 m (1 dp).

Page 7

35. $3x(7 - 2x)$

36. $5x(1 - 3x)$

37. $x^3yz\,(z + y + x)$

38. $3b(4a^2 + a - 3b)$

39. $2\pi r(r + h)$

40. $7km(4k - 7m)$

41. $4k^2(1 - 3k)$

42. $12xy(3x - 2y)$

43. $(k - 3)(4 + y)$

44. $(x + 2)(x + 7)$

45. $(x - 7)(x - 1)$

46. $(x - a)(4 - 2y)$
$= 2(x - a)(2 - y)$

47. $(x + 2)(12x + a)$

48. $(e + 1)(3ac - 6b)$
$= 3(e + 1)(ac - 2b)$

49. $(4x - 1)(2x + 3z)$

Page 8

50. $(x + 6)(x + 1)$

51. $(x + 3)(x + 4)$

52. $x(x + 9)$

53. $(x - 8)(x + 3)$

54. $(a + 6)(a - 4)$

55. $(z + 5)(z - 5)$

56. $(x + 5)(x + 1)$

57. $(x + 8)(x - 3)$

58. $(x - 4)(x - 8)$

59. $(h + 6)(h + 6) = (h + 6)^2$

60. $(p + 1)(p - 1)$

61. $(x + 9)(x - 9)$

62. $(m - 1)(m - 6)$

63. $(x - 9)(x + 5)$

64. $(x + 26)(x - 2)$

Page 9

65. $(m + 7)(m - 2)$

66. $(k - 9)(k - 6)$

67. $(r + 7)(r - 4)$

68. $(a - 9)(a + 2)$

69. $(p - 6)^2$

70. $(n + 1)(n + 12)$

71. $(s - 17)(s + 4)$

72. $(x + 14)(x - 8)$

73. $(k + 12)(k - 9)$

74. $(x + 10)(x - 10)$

75. $(4k - 7)(4k + 7)$

76. $(y - 8x)(y + 8x)$

Page 9 cont...

77. $(5 - 6a)(5 + 6a)$

78. $(1 - pq)(1 + pq)$

79. $(b - \frac{1}{b})(b + \frac{1}{b})$

80. $(x - 10)^2$

81. $(x + 8)^2$

82. $(x + 11)^2$

83. $(x + 13)^2$

84. $(x - 7)^2$

85. $(x - 15)^2$

Page 10

86. $2(x - 4)(x + 1)$

87. $3(x - 3)(x - 1)$

88. $3(x + 10)(x - 1)$

89. $(4b + 5)(b + 2)$

90. $(x - 3)(3x + 2)$

91. $(2n - 5)(3n - 4)$

92. $3(x - 8)(x + 5)$

93. $4(x + 7)(x - 2)$

94. $2(n + 1)(n + 12)$

Page 11

95. $3(p + 7)(p - 7)$

96. $5(q - 7)(q - 3)$

97. $0.5(n - 8)(n + 6)$

98. $(3x + 1)(x + 1)$

99. $(4x + 1)(x - 1)$

100. $(5x + 1)(x - 2)$

101. $(7x + 3)(x + 2)$

102. $(3x - 1)(3x + 4)$

103. $(2x + 3)(3x - 2)$

104. a) Length = $60 - 2w$
Width = $40 - w$

b) $2w^2 - 140w + 2400 = 1408$

$w^2 - 70w + 496 = 0$

c) $w = 8$ m (reject 62)

Length 44 m, Width 32 m

Page 14

105. $\frac{1}{81}$

106. 2

107. $\frac{1}{125}$